PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2026, **60**(3), p. 123–140

Mathematics

ON EPIMORPHISMS OF HYPERGRAPHIC AUTOMATA AND INPUT SYMBOL SEMIGROUPS

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Hypergraphic automata are automata, whose state sets and output symbol sets are hypergraphs invariant under the actions of the transition and output functions. Universally attracting objects in the category of such automata are called universal hypergraphic automata; their semigroups of input symbols are algebras of mappings whose properties are tightly linked to the algebraic structure of the automata themselves. In this paper we describe surjective morphisms (epimorphisms) of universal hypergraphic automata and of their semigroups of input symbols, for effective hypergraphs with p-definable edges. Under two mild hypotheses, a triple $(f,\mathbb{P}_s,g)$ is an epimorphism if and only if the state component f is an isomorphism and the output component g is a homomorphism satisfying a lifting condition. We introduce weak and strong epimorphisms of hypergraphs, show that they differ, and prove that liftability yields strong surjectivity under a size condition on edges. For automata whose state hypergraphs are projective or affine planes, the state components of epimorphisms are exactly the collineations.

https://doi.org/10.46991/PYSUA.2026.60.3.123

MSC2020: 08A35, 08A70.

Keywords: hypergraphic automaton, epimorphism, semigroup, effective hypergraph, p-definable edges, p-hypergraph, projective plane, lifting property.

Introduction. Mappings are composed as presented by B. Plotkin [1]: for $f\colon A \to B$ and $\varphi\colon B \to C$ the product is $a(f\varphi) = (af)\varphi$ for every $a \in A$. From Section "Preliminaries" on we write mappings on the left and compose them as functions. The identity mapping of a set A is ε_A, the set of all mappings from A to B is $\mathrm{Fun}(A,B) = B^A$, and a mapping $f^{-1}\colon B \to A$ is the *inverse* of $f\colon A \to B$, if $ff^{-1} = \varepsilon_A$ and $f^{-1}f = \varepsilon_B$. Thus, f^{-1} exists if and only if f is bijective. A transformation of A is a mapping $A \to A$; the set of all transformations is denoted by S_A.

Following [1], an *automaton* is a multi-sorted algebraic system $\mathcal{A} = (X,S,Y,\delta,\lambda)$, where X is a set of states, Y is a set of output symbols, S is a semigroup of input

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symbols, and $\delta\colon X\times S\to X$, $\lambda\colon X\times S\to Y$ are the transition and output functions satisfying $\delta(x,ab)=\delta(\delta(x,a),b)$ and $\lambda(x,ab)=\lambda(\delta(x,a),b)$. Depending on the application, the sets X and Y may carry the structure of objects of a category $\mathbf{K}$ of algebraic systems, with δ_s and λ_s being morphisms of $\mathbf{K}$ for every $s\in S$. For each pair $A,B\in\mathbf{K}$ the category of automata structured over $\mathbf{K}$ has a universally attracting object, the *universal automaton* $\mathrm{Atm}(A,B)$, whose input symbol semigroup, is $S(A,B)=\mathrm{End}A\times\mathrm{Hom}(A,B)$ [1]. Every automaton with state system A and output system B maps homomorphically into $\mathrm{Atm}(A,B)$, so the algebraic structure of $S(A,B)$ governs the behaviour of the whole class.

In this paper $\mathbf{K}$ is the category **Hgr** of effective hypergraphs with p-definable edges. It contains, in particular, all projective planes and all affine planes with more than four points [2, 3], as well as hypergraphs given by partitions of the vertex set into non-singleton classes. For general background on hypergraphs we refer to [4, 5], and for semigroups, relations and languages of automata to [6–8].

The study of universal hypergraphic automata by means of their input symbol semigroups was started by Molchanov and Khvorostukhina. It is known that $\mathrm{Atm}(H_X,H_Y)$ is determined up to isomorphism by $S(H_X,H_Y)$ [9], that isomorphisms admit a concrete description [10], and, most recently, that monomorphisms admit a complete description [11]. The natural next question, which is the subject of the present paper, is the description of epimorphisms.

Throughout, an *epimorphism* means a *surjective* homomorphism of semigroups, hypergraphs or automata. This is not the categorical notion: for instance, the inclusion $(\mathbb{N},+)\hookrightarrow(\mathbb{Z},+)$ is an epimorphism in the category of semigroups, but is not surjective. Hence all statements below concern surjective morphisms.

For hypergraph homomorphisms the situation requires care, because surjectivity on vertices does not control the edges. We therefore distinguish two notions for a homomorphism $g\colon H_Y\to H_{Y_1}$: g is a *weak* epimorphism, if it is surjective on vertices, and a *strong* epimorphism if, in addition, every edge of H_{Y_1} is the image of an edge of H_Y (see Definition 5). The following small example shows that the notions differ.

Example 1. *Let $p=2$. Let H_Y have vertex set $\{1,\dots,9\}$ and edges $\{1,2,3\},\{4,5,6\},\{7,8,9\}$, and let H_{Y_1} have the same vertices and the same edges together with $\{1,4,7\}$. Both hypergraphs are effective with 2-definable edges (every edge has 3 vertices and distinct edges meet in at most one vertex). The identity map $g=\mathrm{id}$ is a homomorphism $H_Y\to H_{Y_1}$, which is surjective on vertices, hence a weak epimorphism, but the edge $\{1,4,7\}$ is not the image of any edge of H_Y, so g is not a strong epimorphism. Note also that $|L_{Y_1}|=4>|L_Y|=3$.*

A second source of subtlety is the following. Given $\psi_1\in\mathrm{Hom}(H_X,H_{Y_1})$, one wants $\psi\in\mathrm{Hom}(H_X,H_Y)$ with $g\circ\psi=\psi_1$. Surjectivity of g on vertices, or even on edges, does not suffice: a pointwise choice of preimages need not be a homomorphism of hypergraphs (Example 2). Consequently, the description of epimorphisms of $S(H_X,H_Y)$ requires an explicit *lifting condition* on g (Definition 10).

Main results are:

1. We introduce weak and strong epimorphisms of hypergraphs and compare them (Example 1, Proposition 3).

2. We describe when the pair $\mathbb{P}_s = (f^2, f \times g)$ is a well-defined surjective homomorphism $S(H_X, H_Y) \to S(H_{X_1}, H_{Y_1})$. Under two hypotheses (V) and (P) (Theorem), the state component f must be a bijection (in fact an isomorphism) and the output component g must satisfy the lifting condition. Hypothesis (V) cannot be dropped (Example 3) and holds automatically in the geometric case (Lemma 8); the case $p = 1$ is covered only for injective f.

3. We specialise the result to automata whose state hypergraphs are projective or affine planes, where the isomorphisms are collineations (Corollary, Propositions 5 and Propositions 6).

4. We show that epimorphisms transfer recognisable languages through preimages (Proposition 7).

Section "Literature Review" reviews related work. Section "Preliminaries" collects the definitions and the lifting property. Section "Main Results" contains the main theorem. Section gives consequences and applications, and the final section is the Conclusion.

Literature Review. In this section mappings are written on the left, as in the cited sources, and the letters Q and Γ denote the state set and the input alphabet of an automaton; they are local to this section and are not the objects X, Y, S of Section "Introduction".

A (complete, deterministic) automaton is a triple $A = (Q, \Gamma, \delta)$, $\delta\colon Q \times \Gamma \to Q$, where Q is the set of states and Γ the finite input alphabet. The transition function extends uniquely to an action of the free semigroup Γ^+ (equivalently the free monoid Γ^*) on Q by

$$\delta(q, uv) = \delta(\delta(q, u), v), \qquad \delta(q, \varepsilon) = q.$$

This free object is precisely the input symbol semigroup of the automaton. It carries the congruence ρ defined by

$$u \,\rho\, v \iff \forall q \in Q\colon\ \delta(q, u) = \delta(q, v),$$

and the quotient $c(A) = \Gamma^+/\rho$ is known as the characteristic semigroup of A [12]. Equivalently, writing $f_w\colon Q \to Q$, $f_w(q) = \delta(q, w)$, the set $F_A = \{f_w : w \in \Gamma^+\}$ with the product "first f_u, then f_v" is the transformation semigroup of A, and $F_A \cong c(A)$.

A function between state machines $M = (S, I, N)$, $M' = (S', I', N')$ is a homomorphism if it consists of $\gamma\colon S \to S'$, $\iota\colon I \to I'$, with $\gamma(N_1(s, i)) = N_k(\gamma(s), \iota(i))$ [12]. If both γ and ι are onto, M' is a homomorphic image; restricting to $I = I'$ gives endomorphisms, automorphisms, etc. [12]. This is the "input-preserving" notion; requiring ι to be a semigroup homomorphism of the input symbol semigroups yields the strongest form.

Yang reduces the homomorphism problem to closed partitions and output-consistent partitions: an endomorphism of a state machine exists iff there is a non-trivial closed partition (Hartmanis–Stearns), and the homomorphism, isomorphism

and automorphism problems of Moore-type machines are decided by closed, output-consistent partitions [12]. Bavel decomposes an arbitrary finite automaton into its primaries (maximal singly-generated subautomata) and shows that every homomorphism is uniquely a join of homomorphisms on those primaries: any $f \in H(A \to B)$ has a unique decomposition $f = \bigvee_i f_i$, $f_i \in H(A_i \to B)$, over the primaries A_i of A [13].

Hypergraph Semigroups and Graph-algebra Automata. A hypergraph is a triple $H = (V, E, I)$ with $I \subseteq V \times E$ the incidence relation; ordinary graphs, multigraphs and designs are special cases, starting from the groupoid $G(H)$ on $V \times E$ with

$$(v_1, e_1)(v_2, e_2) = \begin{cases} (v_1, e_2), & (v_2, e_1) \in I, \\ \text{undefined}, & \text{otherwise}, \end{cases}$$

one adjoins a zero to obtain $S(H)$, which is isomorphic to the Rees matrix semigroup $M^0(1; E, V; 1)$ with trivial structure group. Two non-empty hypergraphs H_1, H_2 are isomorphic iff $S(H_1) \cong S(H_2)$, and every isomorphism is induced by a uniquely determined hypergraph isomorphism (φ, ψ). Abstractly, a semigroup S is (isomorphic to) a hypergraph semigroup iff it satisfies five conditions:

1) S is categorical at zero;
2) xSy contains at most one non-zero element;
3) if $xSy = \{0\}$, then $xS = \{0\}$ or $Sy = \{0\}$;
4) $st = uv = 0$ forces $s = ux$, $v = xt$ for some x;
5) $S^2 = \{0\}$ or $S^2 = S$.

A graph-algebra automaton is built by using, for $D = (V, E)$ with $E \subseteq V \times V$, the graph algebra $\mathrm{Alg}(D) = V \cup \{0\}$ with product

$$xy = \begin{cases} x, & (x, y) \in E, \\ 0, & \text{otherwise}, \end{cases}$$

as the state set: the automaton $\mathrm{GAtm}(D, T)$ has states $\mathrm{Alg}(D)^1$, initial state 1, terminal set T, and next-state map $a \cdot x = f(x)a$ for a labelling $f \colon \Gamma \to \mathrm{Alg}(D)^1$. (We write GAtm to avoid confusion with the universal automaton $\mathrm{Atm}(A, B)$ of Section "Introduction"). Languages recognised by such automata form a proper subclass of the regular languages, closed under Kleene star and complement.

Homomorphisms of hypergraphs have also been studied from the homotopy point of view: two homomorphisms are weakly homotopic iff they are path-connected in $H^{H'}$, and homotopic in the stronger sense iff their induced poset maps are topologically homotopic [14].

Epimorphisms of Semigroups. A morphism $\alpha \colon S \to T$ is an epimorphism iff for all β, γ with $\alpha\beta = \alpha\gamma$ we have $\beta = \gamma$ [15]. In a concrete category every onto (surjective) morphism is an epimorphism, but the converse fails for semigroups [15]. The obstruction is captured by Isbell's dominion

$$\mathrm{Dom}(U, S) = \{ d \in S : \ \forall \alpha, \beta : S \to T, \ \alpha|_U = \beta|_U \Rightarrow d\alpha = d\beta \}$$

and by Isbell's Zigzag Theorem: $d \in \mathrm{Dom}(U, S)$ iff $d \in U$ or there are $u_0, \ldots, u_{2m} \in U$

and $x_1,\dots,x_m,y_1,\dots,y_m \in S$ with

$$\begin{aligned} &d = u_0 y_1, \quad u_0 = x_1 u_1, \\ &u_{2i-1} y_i = u_{2i} y_{i+1}, \quad x_i u_{2i} = x_{i+1} u_{2i+1} \quad (1 \le i \le m-1), \\ &u_{2m-1} y_m = u_{2m}, \quad x_m u_{2m} = d \end{aligned}$$

see [15]. A subsemigroup U is epimorphically embedded if $\mathrm{Dom}(U,S) = S$, and saturated if it cannot be properly epimorphically embedded [15]. The decisive modern results classify the pseudovarieties V, for which epimorphisms into members are automatically surjective [15]. Every epimorphism to a member of V is onto iff V excludes the 2×2 Brandt aperiodic semigroup B_2; the only finite obstruction to epi-surjectivity from members is the semigroup $Y = B_2 \setminus \{b\}$; precisely three maximal pseudovarieties avoid Y, they are F-saturated, and S (all finite semigroups) is the only F-epimorphically closed pseudovariety containing Y [15].

For a stable semigroup S and a finite subsemigroup U epimorphically embedded in S, every J-maximal element of $S \setminus U$ is regular, and one can find zigzags, whose factors all lie in the D-class of that element [15]. This connects epimorphism theory to the internal structure of the input symbol semigroup, because the Cayley representation realises any finite semigroup as the transition semigroup of a complete semiautomaton (S^1,Γ,δ) with $\delta(s,a) = sa$ [15].

Torsion Theory and Automaton Semigroups. Lam shows a bijective correspondence between hereditary pretorsion classes (HPC) of S-automata, the right linear topologies on the input semigroup S^1, and idempotent preradicals τ on the category of S-automata with $C_\tau = \{M : \tau(M) = M\}$ closed under subautomata and finite products [16]. The correspondence uses the fact that $S^1/\ker(\alpha_x) \cong xS^1$, so that the topology on the input semigroup encodes exactly the quotients allowed [16]. This gives a torsion-theoretic reading of which input-semigroup quotients (i.e. which automaton epimorphisms) preserve membership in a class.

Cain's theory of automaton semigroups takes the input symbol semigroup as intrinsic: an automaton is a deterministic real-time transducer (Q,B,δ), and the generated automaton semigroup $\Sigma(A)$ is the subsemigroup of the tree-endomorphism semigroup $\mathrm{End}\,B^*$ generated by the states [17]. The partial-automaton extension sharpens this: D'Angeli, Rodaro and Wächter show that a semigroup S is a (partial) automaton semigroup iff its structure is compatible with removing zeros from a complete automaton, that S^0 is always a complete automaton semigroup, and that no direct or semidirect product of an arbitrary semigroup with a non-trivial subsemigroup of the free monogenic semigroup is an automaton semigroup [18]. They also prove that automaton-inverse semigroups coincide with inverse automaton semigroups [18].

The results of [15] concern epimorphisms in the categorical sense, which need not be surjective, whereas the works [12,13] treat homomorphisms of automata without hypergraph structure on the state and output sets. In this paper, an epimorphism is a surjective homomorphism, and the objects are the universal hypergraphic automata $\mathrm{Atm}(H_X,H_Y)$ with input symbol semigroups $S(H_X,H_Y)$. Isomorphisms [10] and monomorphisms [11] of such automata have been described before; the description

of epimorphisms is the subject of Sections "Main Results" and "Consequences and Applications".

Preliminaries. From now on mappings are written on the left and composed as functions $(f \circ \varphi)(x) = f(\varphi(x))$. For a mapping φ and a set Z we put $\varphi(Z) = \{\varphi(z) : z \in Z\}$. A constant mapping with value a is denoted C_a, and the identity transformation of X is Δ_X. The notation f^{-1} for a mapping is used *only* when f is bijective; preimages of sets are written $f^{-1}(Z)$ only in the set-theoretic sense and are never used as mappings unless f is a bijection. Here $\mathbb{N} = \{1, 2, \dots\}$.

Definition 1. *A hypergraph is a pair $H = (X, L)$, where $X \neq \varnothing$ is the set of vertices and L is a family of subsets of X called edges. A set $Z \subseteq X$ is bounded in H, if $Z \subseteq l$ for some $l \in L$, and unbounded otherwise. The hypergraph H is effective, if every vertex belongs to at least one edge.*

Definition 2. *Let $p \in \mathbb{N}$. A hypergraph $H = (X, L)$ has p-definable edges, if $|l| \geq p+1$ for every $l \in L$, and any p distinct vertices lie in at most one common edge; equivalently, $|l \cap l'| < p$ for all distinct $l, l' \in L$.*

Definition 3. *Let $p \in \mathbb{N}$. A hypergraph $H = (X, L)$ is a p-hypergraph, if:*

(A1) *any p distinct vertices belong to exactly one common edge;*

(A2) *every edge contains at least $p+1$ vertices;*

(A3) *some $(p+1)$-element subset of X is not bounded in H.*

Every p-hypergraph is effective with p-definable edges. Indeed, by (A3) $|X| \geq p+1$, so every vertex lies in some p-element subset, hence in an edge by (A1); and two distinct edges sharing p vertices would contradict the uniqueness in (A1). Projective planes, and affine planes of order $q \geq 3$, with points as vertices and lines as edges, are 2-hypergraphs [2, 3]. (The affine plane of order 2 has lines with two points and violates (A2).)

Let $H_X = (X, L_X)$ and $H_Y = (Y, L_Y)$ be hypergraphs.

Definition 4. *A mapping $\varphi\colon X \to Y$ is a homomorphism $H_X \to H_Y$, if $\varphi(l)$ is bounded in H_Y for every $l \in L_X$, i.e. $(\forall l \in L_X)(\exists l' \in L_Y)\ \varphi(l) \subseteq l'$. The set of all homomorphisms is $\operatorname{Hom}(H_X, H_Y)$, and $\operatorname{End} H_X = \operatorname{Hom}(H_X, H_X)$. A homomorphism is a monomorphism if it is injective. A bijection $\varphi\colon X \to Y$ is anisomorphism if $l \in L_X \Leftrightarrow \varphi(l) \in L_Y$.*

Compositions of homomorphisms are homomorphisms, so $\operatorname{End} H_X$ is a semigroup under composition. If H_Y is effective, then every mapping $X \to l$ with $l \in L_Y$ is a homomorphism; in particular, all constant mappings C_b, $b \in Y$, belong to $\operatorname{Hom}(H_X, H_Y)$.

The following lemma shows that, for hypergraphs with p-definable edges, a bijection which is a homomorphism in both directions is already an isomorphism.

Lemma 1. *Let H_X, H_Y have p-definable edges and let $\varphi\colon X \to Y$ be a bijection such that both φ and φ^{-1} are homomorphisms. Then φ is an isomorphism.*

Proof. Let $l \in L_X$. Choose $l' \in L_Y$ with $\varphi(l) \subseteq l'$, and then $l'' \in L_X$ with $\varphi^{-1}(l') \subseteq l''$. Then $l \subseteq l''$, so $|l \cap l''| = |l| \geq p+1$, and p-definability gives $l = l''$. Hence, $l \subseteq \varphi^{-1}(l') \subseteq l$, i.e. $\varphi(l) = l' \in L_Y$.

Conversely, let $l' \in L_Y$. Choose $l'' \in L_X$ with $\varphi^{-1}(l') \subseteq l''$ and $l''' \in L_Y$ with $\varphi(l'') \subseteq l'''$. Then $l' \subseteq l'''$, so $l' = l'''$ by p-definability of H_Y, and $\varphi(l'') = l'$. Thus, l' is the image of an edge of H_X. Finally, if $\varphi(l) \in L_Y$ for some $l \subseteq X$, then $\varphi(l) = \varphi(l'')$ for some $l'' \in L_X$ by the previous paragraph, so $l = l'' \in L_X$ by injectivity. □

Weak and Strong Epimorphisms.

Definition 5. *A homomorphism $g\colon H_Y \to H_{Y_1}$ is a weak epimorphism if $g(Y) = Y_1$, and a strong epimorphism if, in addition, $(\forall l_1 \in L_{Y_1})(\exists l \in L_Y)\ g(l) = l_1$.*

Every strong epimorphism is weak. As stated in Section "Introduction", "epimorphism" always means *surjective* homomorphism.

Definition 6. *(Image hypergraph.) For a homomorphism $g\colon H_Y \to H_{Y_1}$ the image hypergraph is $g(H_Y) = (g(Y), \{g(l) : l \in L_Y\})$.*

Even if H_Y and H_{Y_1} have p-definable edges, $g(H_Y)$ need not: g may collapse an edge to fewer than $p+1$ vertices, or the images of two edges may share p or more vertices. Hence $g(H_Y)$ need not belong to the class studied in this paper.

Definition 7. *[1] An automaton is a system $\mathcal{A} = (X, S, Y, \delta, \lambda)$ with S a semigroup and $\delta\colon X \times S \to X$, $\lambda\colon X \times S \to Y$ such that*

$$\delta(x, ab) = \delta(\delta(x,a),b), \qquad \lambda(x,ab) = \lambda(\delta(x,a),b) \qquad (x \in X,\ a,b \in S).$$

Put $\delta_s = \delta(\cdot, s)$ and $\lambda_s = \lambda(\cdot, s)$. It is a hypergraphic automaton, if X and Y carry hypergraph structures H_X, H_Y and $\delta_s \in \mathrm{End}\, H_X$, $\lambda_s \in \mathrm{Hom}(H_X, H_Y)$ for all $s \in S$.

Definition 8. *For hypergraphs H_X, H_Y let $S(H_X, H_Y) = \mathrm{End}\, H_X \times \mathrm{Hom}(H_X, H_Y)$ with multiplication*

$$(\varphi, \psi)(\varphi_1, \psi_1) = (\varphi_1 \circ \varphi,\ \psi_1 \circ \varphi).$$

This is a semigroup, and it is exactly the product $(\varphi\varphi_1, \varphi\psi_1)$ of [1] written in right notation. The universal hypergraphic automaton is $\mathrm{Atm}(H_X, H_Y) = (H_X, S(H_X, H_Y), H_Y, \delta', \lambda')$ with $\delta'(x, (\varphi, \psi)) = \varphi(x)$ and $\lambda'(x, (\varphi, \psi)) = \psi(x)$. For every hypergraphic automaton $\mathcal{A} = (H_X, S, H_Y, \delta, \lambda)$ the map $s \mapsto (\delta_s, \lambda_s)$ is a semigroup homomorphism $S \to S(H_X, H_Y)$ [1].

Morphisms of Universal Automata and the Induced Map.

Definition 9. *Let $f\colon X \to X_1$ and $g\colon Y \to Y_1$ be mappings and $\mathbb{P}_s\colon S(H_X, H_Y) \to S(H_{X_1}, H_{Y_1})$ a semigroup homomorphism. Denote by δ_1', λ_1' the transition and output functions of $\mathrm{Atm}(H_{X_1}, H_{Y_1})$. The triple $\gamma = (f, \mathbb{P}_s, g)$ is compatible, if for all $x \in X$ and $s = (\varphi, \psi)$ with $\mathbb{P}_s(s) = (\chi, \omega)$*

$$f(\delta'(x,s)) = \delta_1'(f(x), \mathbb{P}_s(s)), \qquad g(\lambda'(x,s)) = \lambda_1'(f(x), \mathbb{P}_s(s)),$$

that is,

$$f \circ \varphi = \chi \circ f, \qquad g \circ \psi = \omega \circ f. \tag{1}$$

A compatible triple is a morphism $\mathrm{Atm}(H_X,H_Y) \to \mathrm{Atm}(H_{X_1},H_{Y_1})$ *if* $f \in \mathrm{Hom}(H_X,H_{X_1})$ *and* $g \in \mathrm{Hom}(H_Y,H_{Y_1})$. *A morphism is an epimorphism, if* f, $\mathbb{P}_s$, g *are all surjective, and a strong epimorphism, if moreover* g *is a strong epimorphism of hypergraphs.*

Lemma 2. *If* f *is surjective, then* $\mathbb{P}_s$ *is uniquely determined by* (f,g)*: if* $(f,\mathbb{P}_s,g)$ *and* $(f,\mathbb{P}'_s,g)$ *are compatible, then* $\mathbb{P}_s = \mathbb{P}'_s$. *If* f *is bijective, then necessarily*

$$\mathbb{P}_s(\varphi,\psi) = \big(f\circ\varphi\circ f^{-1},\ g\circ\psi\circ f^{-1}\big). \tag{2}$$

Proof. Let $\mathbb{P}_s(\varphi,\psi) = (\chi,\omega)$ and $\mathbb{P}'_s(\varphi,\psi) = (\chi',\omega')$. By (1), $\chi\circ f = f\circ\varphi = \chi'\circ f$ and $\omega\circ f = g\circ\psi = \omega'\circ f$. A surjective f is right-cancellable, so $\chi=\chi'$ and $\omega=\omega'$. For bijective f compose with f^{-1} on the right. □

For bijective f we therefore write $\mathbb{P}_s = (f^2, f\times g)$, where $f^2(\varphi) = f\circ\varphi\circ f^{-1}$ and $(f\times g)(\psi) = g\circ\psi\circ f^{-1}$. For f merely surjective and not injective, a compatible $\mathbb{P}_s$ defined on all of $S(H_X,H_Y)$ exists only if

$$\ker f \subseteq \ker(f\circ\varphi) \ \text{ and } \ \ker f \subseteq \ker(g\circ\psi) \quad \text{for all } (\varphi,\psi)\in S(H_X,H_Y), \tag{3}$$

because $\chi\circ f$ and $\omega\circ f$ must factor through f. In Section "Main Results" we show that, under hypotheses (V) and (P) of Theorem, the existence of a surjective compatible $\mathbb{P}_s$ (which implies (3)) forces f to be an isomorphism; Example 3 shows that this fails without (V).

Lemma 3. *Let* $f\colon H_X \to H_{X_1}$ *be an isomorphism and* $g\colon H_Y \to H_{Y_1}$ *a homomorphism. Then* (2) *defines a semigroup homomorphism* $\mathbb{P}_s\colon S(H_X,H_Y) \to S(H_{X_1},H_{Y_1})$ *and* $(f,\mathbb{P}_s,g)$ *is compatible.*

Proof. Both f and f^{-1} are homomorphisms. Hence $f\circ\varphi\circ f^{-1}$ is an endomorphism of H_{X_1}, and $g\circ\psi\circ f^{-1}$ is a composition of homomorphisms, so lies in $\mathrm{Hom}(H_{X_1},H_{Y_1})$. For $s=(\varphi,\psi)$, $s_1=(\varphi_1,\psi_1)$,

$$\begin{aligned}\mathbb{P}_s(s)\mathbb{P}_s(s_1) &= \big((f\varphi_1 f^{-1})\circ(f\varphi f^{-1}),\ (g\psi_1 f^{-1})\circ(f\varphi f^{-1})\big)\\ &= \big(f\varphi_1\varphi f^{-1},\ g\psi_1\varphi f^{-1}\big) = \mathbb{P}_s(ss_1).\end{aligned}$$

Compatibility (1) is immediate from (2). □

Lifting. The map $f\times g$ is surjective only, if homomorphisms into H_{Y_1} can be lifted along g to homomorphisms into H_Y. This is a genuine condition and does not follow from surjectivity of g, even on edges.

Definition 10. *(Lifting property). Let* H *be a hypergraph. A homomorphism* $g\colon H_Y \to H_{Y_1}$ *is* H*-liftable, if the map* $\mathrm{Hom}(H,H_Y) \to \mathrm{Hom}(H,H_{Y_1})$, $\psi\mapsto g\circ\psi$, *is surjective.*

Lemma 4. *Let* $f\colon H_X \to H_{X_1}$ *be an isomorphism and* $g\colon H_Y \to H_{Y_1}$ *be a homomorphism. The map* $\psi\mapsto(f\times g)(\psi)$ *from* $\mathrm{Hom}(H_X,H_Y)$ *to* $\mathrm{Hom}(H_{X_1},H_{Y_1})$ *is surjective if and only if* g *is* H_X*-liftable.*

Proof. Since f is an isomorphism, $\psi_1\mapsto\psi_1\circ f$ is a bijection $\mathrm{Hom}(H_{X_1},H_{Y_1}) \to \mathrm{Hom}(H_X,H_{Y_1})$ (its inverse is $\chi\mapsto\chi\circ f^{-1}$), and $(f\times g)(\psi)=\psi_1$ if and only if

$g \circ \psi = \psi_1 \circ f$. Hence every $\psi_1 \in \mathrm{Hom}(H_{X_1}, H_{Y_1})$ is of the form $(f \times g)(\psi)$ if and only if every $\chi \in \mathrm{Hom}(H_X, H_{Y_1})$ is of the form $g \circ \psi$ with $\psi \in \mathrm{Hom}(H_X, H_Y)$, which is H_X-liftability of g. □

Lemma 5. *Let $g\colon H_Y \to H_{Y_1}$ be a homomorphism and let H_{Y_1} be effective. If g is H-liftable for some hypergraph H, then g is a weak epimorphism.*

Proof. For $b \in Y_1$ the constant map C_b is in $\mathrm{Hom}(H, H_{Y_1})$, since H_{Y_1} is effective. A lift ψ satisfies $g \circ \psi = C_b$; taking any vertex x of H gives $b = g(\psi(x)) \in g(Y)$. □

The converse fails, even for strong epimorphisms.

Example 2. *Let $p = 2$. Let H_Y have two disjoint edges $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$. Let H_{Y_1} have vertices $\{a_1, a_2, y, b_1, b_2\}$ and edges $l_1 = \{a_1, a_2, y\}$, $l_2 = \{y, b_1, b_2\}$ (they meet in one vertex). Let g fix a_1, a_2, b_1, b_2 and send $a_3, b_3 \mapsto y$. Then $g(A) = l_1$ and $g(B) = l_2$, so g is a strong epimorphism. Let H_X have vertices x, u_1, u_2, w_1, w_2 and edges $e = \{x, u_1, u_2\}$, $e' = \{x, w_1, w_2\}$. All three hypergraphs are effective with 2-definable edges. Define $\psi_1\colon x \mapsto y$, $u_i \mapsto a_i$, $w_i \mapsto b_i$; then $\psi_1(e) = l_1$ and $\psi_1(e') = l_2$, so $\psi_1 \in \mathrm{Hom}(H_X, H_{Y_1})$. Suppose $g \circ \psi = \psi_1$ for some $\psi \in \mathrm{Hom}(H_X, H_Y)$. Since $\psi(e)$ and $\psi(e')$ are each inside A or B and share the vertex $\psi(x)$, both lie in the same edge, so $g\psi(X) \subseteq l_1$ or $\subseteq l_2$. But $\psi_1(X) \not\subseteq l_1$ and $\psi_1(X) \not\subseteq l_2$ is a contradiction. Hence, the strong epimorphism g is not H_X-liftable.*

Fix $x_0 \in X$ and a subset $F \subseteq Y$ (possibly empty). The language

$$L(x_0, F) = \{(\varphi, \psi) \in S(H_X, H_Y) :\ \psi(x_0) \in F\} \subseteq S(H_X, H_Y)$$

is the set of input symbols that $\mathrm{Atm}(H_X, H_Y)$, started in x_0, maps to an output symbol in F. A subset of $S(H_X, H_Y)$ is *recognisable*, if it equals $L(x_0, F)$ for some x_0 and F. For a *fixed* x_0 the family $\{L(x_0, F)\}_{F \subseteq Y}$ is closed under unions, intersections, and complements: $L(x_0, F) \cup L(x_0, F') = L(x_0, F \cup F')$, $L(x_0, F) \cap L(x_0, F') = L(x_0, F \cap F')$ and $S(H_X, H_Y) \setminus L(x_0, F) = L(x_0, Y \setminus F)$. For different initial states this closure is not claimed.

Problem Statement. Let $\mathrm{Atm}(H_X, H_Y)$ and $\mathrm{Atm}(H_{X_1}, H_{Y_1})$ be universal automata over effective hypergraphs with p-definable edges, and let $f\colon X \to X_1$ be a surjection and $g\colon Y \to Y_1$ be a mapping. We ask for necessary and sufficient conditions on f and g, under which there is a semigroup homomorphism $\mathbb{P}_s$ such that $(f, \mathbb{P}_s, g)$ is a compatible triple with $\mathbb{P}_s$ surjective (such $\mathbb{P}_s$ is unique by Lemma 2, and is given by (2) once f is known to be bijective), and the corresponding conditions for $(f, \mathbb{P}_s, g)$ to be an epimorphism of automata.

Main Results. Throughout this section, $H_X, H_Y, H_{X_1}, H_{Y_1}$ are effective hypergraphs with p-definable edges, $f\colon X \to X_1$ is a surjection, and $g\colon Y \to Y_1$ is a mapping. A triple $(f, \mathbb{P}_s, g)$ is *compatible* in the sense of Definition 9. For a bijective f, Lemma 2 gives $\mathbb{P}_s(\varphi, \psi) = (f\varphi f^{-1}, g\psi f^{-1})$.

Lemma 6. *Let $H = (Z_0, L)$ have p-definable edges and let $Z \subseteq Z_0$ be unbounded with $|Z| \geq p+1$. Then Z contains an unbounded subset with exactly $p+1$ elements. Consequently, every unbounded set contains a finite unbounded subset with at most $p+1$ elements.*

Proof. Suppose every $(p+1)$-subset of Z is bounded. Fix a p-subset $T \subseteq Z$. Since $|Z| \geq p+1$, there is $z \in Z \setminus T$, and $T \cup \{z\}$ lies in an edge; by p-definability the edge r_T containing T is unique. Every $z \in Z \setminus T$ lies in an edge containing T, because $T \cup \{z\}$ is a $(p+1)$-subset of Z, hence in r_T. So, $Z \subseteq r_T$, a contradiction. For the last statement: if $|Z| \leq p$ take Z itself. □

Lemma 7. *Let $V = \{v_1, \ldots, v_{p+1}\}$ have all p-element subsets bounded in H_X. Then every subset of V with at most p elements is bounded in H_X.*

Proof. A subset $S \subseteq V$ with $|S| \leq p$ is contained in a p-element subset T of V, and $T \subseteq l$ for some $l \in L_X$, so $S \subseteq l$. □

Lemma 8. *Suppose H_X satisfies (A1) and H_{X_1} satisfies (A3). Then every surjection $f\colon X \to X_1$ admits a set V as in hypothesis (V) of Theorem.*

Proof. By (A3), there is an unbounded $W \subseteq X_1$ with $|W| = p+1$. Choose $V \subseteq X$ with $f(V) = W$ and $|V| = p+1$ (possible as f is onto: take one preimage of each element of W). By (A1) in H_X, every p-subset of V is bounded, while $f(V) = W$ is unbounded. □

Main Theorem.

Theorem. *Let $f\colon X \to X_1$ be a surjection, and $g\colon Y \to Y_1$ be a mapping. Assume:*

(V) there is $V = \{v_1, \ldots, v_{p+1}\} \subseteq X$ such that every p-element subset of V is bounded in H_X, but $f(V)$ is unbounded in H_{X_1};

(P) $p \geq 2$ (for $p = 1$ one must assume in addition that f is injective).

Then the following are equivalent:

1) there is a surjective semigroup homomorphism $\mathbb{P}_s\colon S(H_X, H_Y) \to S(H_{X_1}, H_{Y_1})$ such that $(f, \mathbb{P}_s, g)$ is compatible;

2) f is an isomorphism $H_X \to H_{X_1}$, and g is a homomorphism which is H_X-liftable (Definition 10);

3) there is $\mathbb{P}_s$ such that $\gamma = (f, \mathbb{P}_s, g)$ is an epimorphism of automata $\mathrm{Atm}(H_X, H_Y) \to \mathrm{Atm}(H_{X_1}, H_{Y_1})$.

In this case g is a weak epimorphism, and $\mathbb{P}_s = (f^2, f \times g)$ is given by (2).

Proof. (3) ⇒ (1) is immediate from the definitions.

(2) ⇒ (1). By Lemma 3, (2) defines a semigroup homomorphism and $(f, \mathbb{P}_s, g)$ is compatible. Surjectivity: let $(\chi, \omega) \in S(H_{X_1}, H_{Y_1})$. Put $\varphi = f^{-1}\chi f$; it is an endomorphism of H_X since f, f^{-1}, χ are homomorphisms. The map ωf lies in $\mathrm{Hom}(H_X, H_{Y_1})$, so by H_X-liftability there is $\psi \in \mathrm{Hom}(H_X, H_Y)$ with $g\psi = \omega f$. Then $\mathbb{P}_s(\varphi, \psi) = (f\varphi f^{-1}, g\psi f^{-1}) = (\chi, \omega)$.

(1) $\Rightarrow$ (2). Let $\mathbb{P}_s$ be as in (1). For $(\varphi,\psi)\in S(H_X,H_Y)$ write $\mathbb{P}_s(\varphi,\psi)=(\chi,\omega)$; compatibility reads

$$f\varphi=\chi f,\qquad g\psi=\omega f. \tag{4}$$

Recall that all constant maps are homomorphisms, since the hypergraphs are effective. The proof proceeds in six steps: Steps 1–4 show that f is an isomorphism, Step 5 show that g is a homomorphism, and Step 6 show that g is H_X-liftable.

Step 1: f is injective. If $p=1$ this is assumed, so let $p\geq 2$. Suppose $f(a)=f(b)$ with $a\neq b$. Since $f(V)$ is unbounded and H_{X_1} is effective, f is not constant on V; pick $v_i,v_j\in V$ with $f(v_i)\neq f(v_j)$. Define $\varphi\colon X\to X$ by $\varphi(a)=v_i$ and $\varphi(x)=v_j$ for $x\neq a$. Then $\varphi(X)\subseteq\{v_i,v_j\}$, which is bounded by Lemma 7 (here $p\geq 2$ is used), so $\varphi\in\operatorname{End}H_X$. Apply (4) to (φ,C_d), with $d\in Y$ arbitrary: $f\varphi=\chi f$, hence $f\varphi(a)=\chi(f(a))=\chi(f(b))=f\varphi(b)$, i.e. $f(v_i)=f(v_j)$, a contradiction. Thus f is a bijection and, by Lemma 2, $\mathbb{P}_s$ is given by (2); in particular

$$f\varphi f^{-1}\in\operatorname{End}H_{X_1}\quad\text{for all }\varphi\in\operatorname{End}H_X. \tag{5}$$

Step 2: f is a homomorphism. Suppose $l\in L_X$ and $f(l)$ is unbounded in H_{X_1}. As $|f(l)|=|l|\geq p+1$, Lemma 6 gives an unbounded $W_1\subseteq f(l)$ with $|W_1|=p+1$. Fix an edge $e_1\in L_{X_1}$ and a map $\chi\colon X_1\to W_1$ with $\chi(e_1)=W_1$ (possible, since $|e_1|\geq p+1$). Put $\varphi=f^{-1}\chi f$. Then $\varphi(X)\subseteq f^{-1}(W_1)\subseteq l$, so $\varphi\in\operatorname{End}H_X$, whereas $f\varphi f^{-1}=\chi$ maps e_1 onto the unbounded set W_1. This contradicts (5).

Step 3: f^{-1} is a homomorphism. Suppose $e_1\in L_{X_1}$ and $Y_0:=f^{-1}(e_1)$ is unbounded in H_X; then $|Y_0|\geq p+1$. We construct $\varphi\in\operatorname{End}H_X$ and distinct $y_1,\dots,y_{p+1}\in Y_0$ with $\varphi(y_i)=v_i$. Then $f\varphi f^{-1}(e_1)\supseteq f(V)$ is unbounded, contradicting (5).

Subcase A: every p-subset of Y_0 is unbounded in H_X. Then $|r\cap Y_0|\leq p-1$ for every edge r. Take any distinct $y_1,\dots,y_{p+1}\in Y_0$ and put $\varphi(y_i)=v_i$ for $i\leq p$ and $\varphi(x)=v_{p+1}$, otherwise. For an edge r, $\varphi(r)\subseteq\{v_i : y_i\in r\}\cup\{v_{p+1}\}$ has at most p elements, hence is bounded by Lemma 7.

Subcase B: some p-subset $\{y_1,\dots,y_p\}\subseteq Y_0$ lies in an edge r_0. Since Y_0 is unbounded, there is $y_{p+1}\in Y_0\setminus r_0$. Define

$$\varphi(x)=\begin{cases} v_i, & x=y_i,\ i\leq p-1,\\ v_p, & x\in r_0\setminus\{y_1,\dots,y_{p-1}\},\\ v_{p+1}, & x\notin r_0.\end{cases}$$

Then $\varphi(y_p)=v_p$ and $\varphi(y_{p+1})=v_{p+1}$. Also $\varphi(r_0)\subseteq\{v_1,\dots,v_p\}$ is bounded. For an edge $r'\neq r_0$ we have $|r'\cap r_0|\leq p-1$ by p-definability, so $\varphi(r')$ consists of at most $p-1$ values from $\{v_1,\dots,v_p\}$ together with possibly v_{p+1}: at most p elements, bounded by Lemma 7. Hence $\varphi\in\operatorname{End}H_X$.

Step 4: f is an isomorphism. By Steps 1–3, f is a bijection and f,f^{-1} are homomorphisms; apply Lemma 1.

Step 5: g is a homomorphism. Suppose $m\in L_Y$ and $g(m)$ is unbounded. By Lemma 6 choose a finite unbounded $T\subseteq g(m)$ with $|T|\leq p+1$, and $u_t\in m$ with

$g(u_t) = t$ for $t \in T$. Fix $r \in L_X$ and let $\psi\colon X \to m$ map r onto $\{u_t : t \in T\}$ (note $|r| \geq p+1 \geq |T|$). Then $\psi \in \mathrm{Hom}(H_X, H_Y)$, and by (4) applied to (Δ_X, ψ), we get $\omega = g\psi f^{-1} \in \mathrm{Hom}(H_{X_1}, H_{Y_1})$. But $f(r) \in L_{X_1}$ and $\omega(f(r)) = g\psi(r) = T$ is unbounded, a contradiction.

Step 6: g is H_X-liftable. Let $\psi_1 \in \mathrm{Hom}(H_X, H_{Y_1})$. Then $\omega_1 = \psi_1 f^{-1} \in \mathrm{Hom}(H_{X_1}, H_{Y_1})$, and $(\Delta_{X_1}, \omega_1) \in S(H_{X_1}, H_{Y_1})$. By surjectivity of $\mathbb{P}_s$, there is (φ, ψ) with $\mathbb{P}_s(\varphi, \psi) = (\Delta_{X_1}, \omega_1)$, i.e. $g\psi f^{-1} = \omega_1$, i.e. $g\psi = \psi_1$.

(2) $\Rightarrow$ (3). By (2) $\Rightarrow$ (1) we have a surjective $\mathbb{P}_s$ with $(f, \mathbb{P}_s, g)$ compatible; f and g are homomorphisms, f is bijective, and g is surjective by Lemma 5. The last statement of the Theorem follows from Lemma 5 and Lemma 2. □

Example 3. *Let $p \geq 1$ and $m \geq p+1$. Let H_X consist of m pairwise disjoint edges $r_1, \ldots, r_m$ of size $p+1$, and let H_{X_1} have $X_1 = \{1, \ldots, m\}$ and the single edge X_1. Let $f(x) = k$ for $x \in r_k$. Similarly let H_Y consist of m disjoint edges $s_1, \ldots, s_m$ of size $p+1$, H_{Y_1} have vertex set $\{1, \ldots, m\}$ and the single edge Y_1, and let $g(y) = k$ for $y \in s_k$. All four hypergraphs are effective with p-definable edges. Since X_1 and Y_1 are themselves edges,* every *mapping $X_1 \to X_1$ is an endomorphism and every mapping $X_1 \to Y_1$ is a homomorphism.*

• For $(\varphi, \psi) \in S(H_X, H_Y)$ we have $\varphi(r_k) \subseteq r_{k'}$ and $\psi(r_k) \subseteq s_{k''}$ for some k', k'', so $f\varphi$ and $g\psi$ are constant on each r_k. Hence $f\varphi = \chi f$ and $g\psi = \omega f$ for unique χ, ω (f is onto), and $(\chi, \omega) \in S(H_{X_1}, H_{Y_1})$. Put $\mathbb{P}_s(\varphi, \psi) = (\chi, \omega)$. If $(\varphi_1, \psi_1) \mapsto (\chi_1, \omega_1)$, then $f\varphi_1\varphi = \chi_1 f\varphi = \chi_1\chi f$ and $g\psi_1\varphi = \omega_1 f\varphi = \omega_1\chi f$, so by uniqueness $\mathbb{P}_s\big((\varphi, \psi)(\varphi_1, \psi_1)\big) = (\chi_1\chi, \omega_1\chi) = \mathbb{P}_s(\varphi, \psi)\mathbb{P}_s(\varphi_1, \psi_1)$; thus $\mathbb{P}_s$ is a semigroup homomorphism.

• $\mathbb{P}_s$ is surjective: for (χ, ω) let φ be constant on each r_k with value in $f^{-1}(\chi(k))$, and ψ constant on r_k with value in $g^{-1}(\omega(k))$. Then $\varphi \in \mathrm{End}\, H_X$, $\psi \in \mathrm{Hom}(H_X, H_Y)$ and $\mathbb{P}_s(\varphi, \psi) = (\chi, \omega)$.

Moreover f and g are surjective homomorphisms. Thus $(f, \mathbb{P}_s, g)$ is an epimorphism of automata although f is not injective. Condition (P) holds when $p \geq 2$, and (V) fails, because every subset of X_1 is bounded in H_{X_1}.

By Lemma 5 and Example 2, H_X-liftability is strictly stronger than being a weak, and even a strong, epimorphism. The following gives a simple sufficient condition.

Proposition 1. *If $g\colon H_Y \to H_{Y_1}$ has a homomorphic section $s\colon H_{Y_1} \to H_Y$ (i.e. $s \in \mathrm{Hom}(H_{Y_1}, H_Y)$ and $g \circ s = \Delta_{Y_1}$), then g is H-liftable for every hypergraph H.*

Proof. Let H be a hypergraph and $\psi_1 \in \mathrm{Hom}(H, H_{Y_1})$, and put $\psi = s \circ \psi_1\colon X(H) \to Y$. We check that ψ is a homomorphism and that it lifts ψ_1.

Let $e \in L_H$. Since ψ_1 is a homomorphism, there is $l_1 \in L_{Y_1}$ with $\psi_1(e) \subseteq l_1$. Since s is a homomorphism, there is $l \in L_Y$ with $s(l_1) \subseteq l$. Hence $\psi(e) = s(\psi_1(e)) \subseteq s(l_1) \subseteq l$, so $\psi(e)$ is bounded in H_Y, i.e. $\psi \in \mathrm{Hom}(H, H_Y)$. By associativity of composition, $g \circ \psi = (g \circ s) \circ \psi_1 = \Delta_{Y_1} \circ \psi_1 = \psi_1$.

Since ψ_1 was arbitrary, $\psi \mapsto g \circ \psi$ is a surjection $\mathrm{Hom}(H, H_Y) \to \mathrm{Hom}(H, H_{Y_1})$, i.e. g is H-liftable. In particular every isomorphism g is liftable. □

Proposition 2. *For a homomorphism $g\colon H_Y \to H_{Y_1}$ the following are equivalent:*

1. *g has a homomorphic section;*

2. *g is H-liftable for every hypergraph H;*

3. *g is H_{Y_1}-liftable.*

Proof. (1) $\Rightarrow$ (2) is Proposition 1. (2) $\Rightarrow$ (3) is trivial. (3) $\Rightarrow$ (1): apply liftability to $\psi_1 = \Delta_{Y_1} \in \mathrm{Hom}(H_{Y_1}, H_{Y_1})$ to get $s \in \mathrm{Hom}(H_{Y_1}, H_Y)$ with $g \circ s = \Delta_{Y_1}$. □

Corollary. *Let H_X, H_{X_1} be projective planes or affine planes of order $q \geq 3$, viewed as 2-hypergraphs, and let H_Y, H_{Y_1} be effective with 2-definable edges. Let $f\colon X \to X_1$ be a surjection and $g\colon Y \to Y_1$ a mapping. The following are equivalent:*

1) there is a surjective $\mathbb{P}_s$ with $(f, \mathbb{P}_s, g)$ compatible;

2) f is a collineation and g is an H_X-liftable homomorphism;

3) $(f, \mathbb{P}_s, g)$ is an epimorphism of automata for some $\mathbb{P}_s$.

Proof. Let H be a projective plane, or an affine plane of order $q \geq 3$. Then

(A1) any two points lie on exactly one common line;

(A2) every line has at least 3 points: it has $q+1 \geq 3$ points in the projective case and $q \geq 3$ in the affine case;

(A3) there are three non-collinear points, i.e. a 3-element set not contained in any line.

Hence H is a 2-hypergraph, and it is effective with 2-definable edges. Thus H_X and H_{X_1}, as well as H_Y and H_{Y_1} by assumption, satisfy the standing assumptions of Section with $p = 2$.

Condition (P) holds since $p = 2 \geq 2$. Since H_X satisfies (A1) and H_{X_1} satisfies (A3), Lemma 8 gives a set $V \subseteq X$ with $|V| = 3$ such that every 2-element subset of V is bounded in H_X, while $f(V)$ is unbounded in H_{X_1} (three non-collinear points). So (V) holds for every surjection f, and Theorem applies. By Definition 4, f is an isomorphism of hypergraphs iff f is a bijection with $l \in L_X \Leftrightarrow f(l) \in L_{X_1}$, i.e. iff f is a collineation. Replacing "isomorphism" by "collineation" in Theorem, (2) gives the claim. If H_X is projective and H_{X_1} is affine, or vice versa, no collineation exists: a collineation induces a bijection between the sets of lines and preserves the property of two lines having a common point, while in a projective plane any two lines meet and an affine plane has parallel lines. Then (2) fails for every f, and by Theorem so do (1) and (3). □

Consequences and Applications. Throughout this section all hypergraphs are effective with p-definable edges, and a morphism $\gamma = (f, \mathbb{P}_s, g)$ is compatible in the sense of Definition 9. Hypotheses (V) and (P) of Theorem are in force whenever that theorem is invoked.

Theorem describes the output component g by liftability. The next result shows that liftability recovers strong surjectivity on edges under a mild size condition, and that without liftability the gap between weak and strong epimorphisms is real.

Proposition 3. *Let $g\colon H_Y \to H_{Y_1}$ be an H_X-liftable homomorphism. Let $l_1 \in L_{Y_1}$ and suppose that*

$$|e| \geq |l_1| \text{ for some } e \in L_X. \tag{6}$$

Then $l_1 = g(l)$ for some $l \in L_Y$. In particular, if (6) holds for every edge l_1 of H_{Y_1}, then g is a strong epimorphism.

Proof. Choose $e \in L_X$ with $|e| \geq |l_1|$ and a mapping $\psi_1\colon X \to l_1$ with $\psi_1(e) = l_1$ (possible since $|e| \geq |l_1|$). As $\psi_1(X) \subseteq l_1$, we have $\psi_1 \in \mathrm{Hom}(H_X, H_{Y_1})$. By liftability, there is $\psi \in \mathrm{Hom}(H_X, H_Y)$ with $g\psi = \psi_1$. Let $l \in L_Y$ with $\psi(e) \subseteq l$. Then $l_1 = \psi_1(e) = g(\psi(e)) \subseteq g(l)$. Since g is a homomorphism, $g(l) \subseteq l_1'$ for some $l_1' \in L_{Y_1}$, so $l_1 \subseteq l_1'$ and $|l_1 \cap l_1'| = |l_1| \geq p+1$. By p-definability, $l_1 = l_1'$. Hence $g(l) = l_1$.

If (6) holds for every edge of H_{Y_1}, then every edge of H_{Y_1} is the image of an edge of H_Y; moreover g is a weak epimorphism by Lemma 5. Thus g is a strong epimorphism. □

For example, let $g = \mathrm{id}\colon H_Y \to H_{Y_1}$ be the identity of Example 1; it is a weak epimorphism that is not strong. It is also not H_X-liftable for any effective H_X with $p = 2$. Indeed, take an edge e of H_X (so $|e| \geq 3$) and $\psi_1\colon X \to \{1,4,7\}$ with $\psi_1(e) = \{1,4,7\}$; then $\psi_1 \in \mathrm{Hom}(H_X, H_{Y_1})$, and $g\psi = \psi_1$ forces $\psi = \psi_1$, but $\psi_1(e)$ is not bounded in H_Y. Hence, by Theorem, this g is not the output component of an epimorphism of automata.

The size condition (6) cannot be dropped altogether.

Example 4. *Let $p = 2$. Let H_X have the vertex set $X = \{x_1, x_2, x_3\}$ and the single edge X. Let H_{Y_1} have the vertex set $Y_1 = \{1,2,3,4\}$ and the single edge Y_1. Let H_Y consist of four pairwise disjoint edges $m_i = \{y_{i,j} : j \in Y_1 \setminus \{i\}\}$, $i \in Y_1$, and put $g(y_{i,j}) = j$. All three hypergraphs are effective with 2-definable edges, and g maps m_i bijectively onto $Y_1 \setminus \{i\}$. Hence g is a homomorphism, it is surjective on vertices, but $g(m_i) \neq Y_1$ for every i, so g is a weak epimorphism that is not strong. On the other hand g is H_X-liftable: every mapping $\psi_1\colon X \to Y_1$ is a homomorphism (since Y_1 is an edge), and $|\psi_1(X)| \leq 3$, so $\psi_1(X) \subseteq Y_1 \setminus \{i\}$ for some i; then $\psi(x_k) = y_{i,\psi_1(x_k)}$ defines $\psi\colon X \to m_i$, a homomorphism with $g\psi = \psi_1$. Here $|e| = 3 < 4 = |Y_1|$, so (6) fails.*

Proposition 4. *Assume (V) and (P). Let $\gamma = (f, \mathbb{P}_s, g)\colon \mathrm{Atm}(H_X, H_Y) \to \mathrm{Atm}(H_{X_1}, H_{Y_1})$ be an epimorphism. Then:*

1) *$f\colon H_X \to H_{X_1}$ is an isomorphism; in particular $|X| = |X_1|$ and $|L_X| = |L_{X_1}|$;*
2) *g is a surjective H_X-liftable homomorphism, so $|Y_1| \leq |Y|$;*
3) *if g is a strong epimorphism, then $|L_{Y_1}| \leq |L_Y|$.*

Proof. (1) and (2) follow from Theorem, (3) ⇒ (2), and the definition of an epimorphism (which includes surjectivity of g); an isomorphism preserves the

numbers of vertices and edges. For (3), if every edge of H_{Y_1} is the image of an edge of H_Y, choose for each $l_1 \in L_{Y_1}$ an edge $l \in L_Y$ with $g(l) = l_1$; distinct l_1 give distinct l, so $|L_{Y_1}| \leq |L_Y|$. □

The inequality $|L_{Y_1}| \leq |L_Y|$ is not valid for weak epimorphisms: in Example 1, $|L_{Y_1}| = 4 > 3 = |L_Y|$. In words: the state component of an epimorphism of universal hypergraphic automata never identifies states, and all compression happens on the output side, where g must in addition be liftable. This contrasts with general automata, where an epimorphism may identify states. We call the property that f is necessarily an isomorphism *state rigidity*.

Recall that for a prime power $q = r^e$ the projective plane $\mathrm{PG}(2,q)$ has q^2+q+1 points and lines of size $q+1$, and its collineation group is $\mathrm{P\Gamma L}(3,q)$, of order $e\,q^3(q^3-1)(q^2-1)$. The affine plane $\mathrm{AG}(2,q)$ has q^2 points and lines of size q. Its collineation group is $\mathrm{A\Gamma L}(2,q)$, of order $e\,q^2(q^2-1)(q^2-q)$ [2, 3]. Both are 2-hypergraphs, $\mathrm{PG}(2,q)$ for every q and $\mathrm{AG}(2,q)$ for $q \geq 3$; for $\mathrm{PG}(2,2)$ axiom (A3) is witnessed by a non-collinear triple. By Lemma 8 hypothesis (V) holds automatically and $p = 2$ satisfies (P).

Proposition 5. *Let $H_X = H_{X_1} = \mathrm{PG}(2,q)$ and let H_Y, H_{Y_1} be effective with 2-definable edges. Let $f\colon X \to X_1$ be a surjection and $g\colon Y \to Y_1$ a mapping. Then $(f, \mathbb{P}_s, g)$ is an epimorphism of automata for some $\mathbb{P}_s$ if and only if*

1. *$f \in \mathrm{P\Gamma L}(3,q)$ and*
2. *g is an H_X-liftable homomorphism.*

In that case, if moreover every edge of H_{Y_1} has at most $q+1$ vertices, then g is a strong epimorphism. Every $f \in \mathrm{P\Gamma L}(3,q)$ occurs as the state component of an epimorphism: take $H_{Y_1} = H_Y$ and $g = \Delta_Y$.

Proof. The equivalence is Corollary, since the collineations of $\mathrm{PG}(2,q)$ are the elements of $\mathrm{P\Gamma L}(3,q)$. The bound on edge sizes gives (6) for every edge of H_{Y_1}, because the lines of H_X have $q+1$ points; Proposition 3 then applies. For the last statement, Δ_Y is an isomorphism, hence liftable by Proposition 1, and Corollary applies. □

Proposition 6. *Let $H_X = H_{X_1} = \mathrm{AG}(2,q)$ with $q \geq 3$. Then the statement of Proposition 5 holds with $\mathrm{P\Gamma L}(3,q)$ replaced by $\mathrm{A\Gamma L}(2,q)$ and "$q+1$" replaced by "q".*

For $q = 3$, $|\mathrm{P\Gamma L}(3,3)| = 3^3 \cdot 26 \cdot 8 = 5616$ and $|\mathrm{A\Gamma L}(2,3)| = 3^2 \cdot 8 \cdot 6 = 432$. Thus, for every H_Y, the automaton $\mathrm{Atm}(\mathrm{PG}(2,3), H_Y)$ has exactly 5616 possible state components of epimorphisms into automata with the same state hypergraph.

Let $L(x_0, F)$ be as in Section "Preliminaries", where $x_0 \in X$ and $F \subseteq Y$, and let $L_1(x_1, F_1)$ be defined likewise in $S(H_{X_1}, H_{Y_1})$, for $x_1 \in X_1$ and $F_1 \subseteq Y_1$.

Proposition 7. *Let $\gamma = (f, \mathbb{P}_s, g)$ be a compatible triple, $x_0 \in X$ and $F_1 \subseteq Y_1$. Then*

$$\mathbb{P}_s^{-1}\big(L_1(f(x_0), F_1)\big) = L\big(x_0, g^{-1}(F_1)\big). \tag{7}$$

If γ is an epimorphism of automata, then

1. $\mathbb{P}_s\big(L(x_0, g^{-1}(F_1))\big) = L_1(f(x_0), F_1)$, *and every recognisable language of* $\mathrm{Atm}(H_{X_1}, H_{Y_1})$ *is the image under* $\mathbb{P}_s$ *of a recognisable language of* $\mathrm{Atm}(H_X, H_Y)$*;*

2. *for every* $F \subseteq Y$, $\mathbb{P}_s(L(x_0, F)) \subseteq L_1(f(x_0), g(F))$, *with equality when* $F = g^{-1}(g(F))$ *(i.e. F is a union of fibres of g).*

Proof. For $s = (\varphi, \psi)$ with $\mathbb{P}_s(s) = (\chi, \omega)$, compatibility gives $\omega(f(x_0)) = g(\psi(x_0))$. Hence $\mathbb{P}_s(s) \in L_1(f(x_0), F_1)$ iff $g(\psi(x_0)) \in F_1$, iff $\psi(x_0) \in g^{-1}(F_1)$, iff $s \in L(x_0, g^{-1}(F_1))$. This is (7). If $\mathbb{P}_s$ is surjective, then $\mathbb{P}_s(\mathbb{P}_s^{-1}(A)) = A$ for every A, which gives the first statement of (1); every initial state of the target has the form $f(x_0)$ since f is surjective, which gives the second. For (2), $s \in L(x_0, F)$ gives $\omega(f(x_0)) = g(\psi(x_0)) \in g(F)$; for $F = g^{-1}g(F)$ apply (1) with $F_1 = g(F)$. □

Thus, the transfer of recognisable languages along an epimorphism of automata is described by (7) and Proposition 7.

Conclusion. We described surjective morphisms between universal hypergraphic automata $\mathrm{Atm}(H_X, H_Y)$ and between their input symbol semigroups $S(H_X, H_Y)$ for effective hypergraphs with p-definable edges. Under hypotheses (V) and (P), a triple (f, P_s, g) is an epimorphism of automata if and only if the state component f is an isomorphism (state rigidity) and the output component g is a homomorphism which is H_X-liftable (Theorem). Hypothesis (V) cannot be dropped (Example 3); it holds automatically when H_X satisfies (A1) and H_{X_1} satisfies (A3) (Lemma 8).

Weak and strong epimorphisms of hypergraphs differ (Example 1). Liftability implies weak surjectivity (Lemma 5), but is not implied even by strong surjectivity (Example 2); it forces strong surjectivity on every edge of H_{Y_1} not larger than some edge of H_X (Proposition 3), and this size condition cannot be dropped (Example 4). For projective and affine planes the state components of epimorphisms are exactly the collineations (Corollary). Epimorphisms transfer recognisable languages through preimages (Proposition 7). The results of [9–11] concern the determination of $\mathrm{Atm}(H_X, H_Y)$ by $S(H_X, H_Y)$, isomorphisms and monomorphisms of universal hypergraphic automata; the present paper treats surjective morphisms under (V) and (P).

Received 12.10.2026
Reviewed 04.11.2026
Accepted 26.11.2026

REFERENCES

1. Plotkin B.I., Greenglaz L.Ja., Gvaramija A.A. *Algebraic Structures in Automata and Databases Theory*. Singapore, World Scientific (1992).
2. Hartshorne R. *Foundations of Projective Geometry*. New York, Ishi Press (2009) (reprint of the 1967 edition).

3. Kárteszi F. *Introduction to Finite Geometries*. Amsterdam, North-Holland (1976).
4. Bretto A. *Hypergraph Theory: An Introduction*. Springer, Cham (2013).
5. Zykov A.A. Hypergraphs. *Uspekhi Mat. Nauk* **29** (1974), 89–154.
6. Eilenberg S. *Automata, Languages, and Machines* **V. A.** New York, Academic Press (1974).
7. Lallement G. *Semigroups and Combinatorial Applications*. New York, Wiley (1979).
8. Vagner V.V. *The Theory of Semigroups and Their Applications*. **V. 1.** Saratov, Saratov University (1965).
9. Molchanov V.A., Khvorostukhina E.V. On Problem of Abstract Definability of Universal Hypergraphic Automata by Input Symbol Semigroup. *Chebyshevskii Sb.* **20** (2019), 251–264.
https://doi.org/10.18500/1816-9791-2022-22-3-293-306
10. Khvorostukhina E.V., Molchanov V.A. On Problem of Concrete Characterization of Universal Automata. *Lobachevskii J. Math.* **38** (2017), 664–669.
https://doi.org/10.1134/S1995080217040114
11. Khvorostukhina E.V. On Monomorphisms of Universal Hypergraphic Automata. *Lobachevskii J. Math.* **46** (2025), 6143–6149.
https://doi.org/10.1134/S1995080225614109
12. Yang C.-C. Structural Preserving Morphisms of Finite Automata and an Application to Graph Isomorphism. *IEEE Trans. Comput.* **C-24** (1975), 1133–1139.
https://doi.org/10.1109/t-c.1975.224148
13. Bavel Z. Structure and Transition-Preserving Functions of Finite Automata. *J. ACM* **15** (1968), 135–158.
https://doi.org/10.1145/321439.321448
14. Zhang C., et al. Homotopy and Hom Construction in the Category of Finite Hypergraphs. *Graphs Combin.* **39** (2023).
https://doi.org/10.1007/s00373-023-02672-6
15. Almeida J., Shah A.H. Epimorphisms and Pseudovarieties of Semigroups. *Semigroup Forum* (2025).
https://doi.org/10.1007/s00233-025-10600-8
16. Lam C.S. Some Characterizations of the Hereditary Pretorsion Class of Semigroup Automata. *Glasgow Math. J.* **35** (1993), 327–337.
https://doi.org/10.1017/s0017089500009903
17. Cain A.J. Automaton Semigroups. *Theoret. Comput. Sci.* **410** (2009), 5022–5038.
https://doi.org/10.1016/j.tcs.2009.07.054
18. D'Angeli D., RodaroE., Wächter J.P. On the Structure Theory of Partial Automaton Semigroups. *Semigroup Forum* **101** (2020), 51–76.
https://doi.org/10.1007/s00233-020-10114-5

ԺԱՍԵՄ ՀԱՄՈՒԴ

ՀԻՊԵՐԳՐԱՖԻԿ ԱՆՏՈՄԱՏՆԵՐԻ ԵՎ ՄՈՒՏՔԱՅԻՆ ՍԻՄՎՈՐԼԵՐԻ ԿԻՍԱԽՄԲԵՐԻ ԷՊԻՄՈՐՖԻԶՆԵՐԻ ՄԱՍԻՆ

Հիպերգրաֆիկ ավտոմատները ավտոմատներ են, որոնց վիճակի բազմությունները և ելքային սիմվոլների բազմությունները անփոփոխ հիպերգրաֆներ են անցումային և ելքային ֆունկցիաների ազդեցության տակ: Նման ավտոմատնե-

րի կատեգորիայի համընդհանուր գրավիչ օբյեկտները կոչվում են համընդհանուր հիպերգրաֆիկ ավտոմատներ. դրանց մուտքային սիմվոլների կիսախմբերը քարտեզագրող հանրահաշիվներ են, որոնց հատկությունները սերտորեն կապված են ավտոմատների հանրահաշվական կառուցվածքի հետ: Այս աշխատանքում մենք նկարագրում ենք համընդհանուր հիպերգրաֆիկ ավտոմատների սյուրյեկտիվ մորֆիզմները (էպիմորֆիզմները) և դրանց մուտքային սիմվոլների կիսախմբերը պ-որոշելի կողերով արդյունավետ հիպերգրաֆների համար: Երկու մեղմ վարկածների դեպքում՝ եռակի $(f,\mathbb{P}_s,g)$-ը էպիմորֆիզմ է միայն այն դեպքում, եթե f վիճակի բաղադրիչը իզոմորֆիզմ է, իսկ g ելքային բաղադրիչը հոմոմորֆիզմ է, որը բավարարում է բարձրացման պայմանը: Մենք ներկայացնում ենք հիպերգրաֆների թույլ և ուժեղ էպիմորֆիզմները, ցույց ենք տալիս, որ դրանք տարբեր են և ապացուցում ենք, որ բարձրացման հնարավորությունը ենթադրում է ուժեղ սյուրյեկտիվություն կողերի չափերի պայմաններում: Այն ավտոմատների համար, որոնց վիճակի հիպերգրաֆները պրոյեկտիվ կամ աֆինային հարթություններ են, էպիմորֆիզմների վիճակի բաղադրիչները հենց կոլլինեացիաներն են:

ЖАСЕМ ХАМУД

ОБ ЭПИМОРФИЗМАХ ГИПЕРГРАФИЧЕСКИХ АВТОМАТОВ И ПОЛУГРУПП ВХОДНЫХ СИМВОЛОВ

Гиперграфические автоматы – это автоматы, множества состояний и множества выходных символов которых являются гиперграфами, инвариантными относительно действий функций переходов и выходов. Универсально притягивающие объекты в категории таких автоматов называются универсальными гиперграфическими автоматами; их полугруппы входных символов представляют собой алгебры отображений, свойства которых тесно связаны с алгебраической структурой самих автоматов. В настоящей работе мы описываем сюръективные морфизмы (эпиморфизмы) универсальных гиперграфических автоматов и их полугрупп входных символов для эффективных гиперграфов с p-определимыми ребрами. При двух мягких гипотезах тройка $(f,\mathbb{P}_s,g)$ является эпиморфизмом тогда и только тогда, когда компонента состояний f является изоморфизмом, а компонента выходов g – гомоморфизмом, удовлетворяющим условию подъема. Мы вводим слабые и сильные эпиморфизмы гиперграфов, показываем, что они различны, и доказываем, что поднимаемость влечет сильную сюръективность при условии на размер ребер. Для автоматов, гиперграфы состояний которых являются проективными или аффинными плоскостями, компоненты состояний эпиморфизмов в точности являются коллинеациями.