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Mathematics

HERMITE MULTIVARIATE INTERPOLATION FORMULA

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We present a new formula for the Hermite multivariate interpolation problem in the framework of the Chung–Yao approach.

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Chung–Yao Lagrange Interpolation. For $x = (x_1, \dots, x_k)$, $y = (y_1, \dots, y_k) \in \mathbb{R}^k$ and multi-index $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{Z}^k_{\geq 0}$, we adopt the standard multi-index notation:

$$x \cdot y = \sum_{i=1}^{k} x_i y_i, \qquad x^\alpha = \prod_{i=1}^{k} x_i^{\alpha_i}, \qquad |\alpha| = \sum_{i=1}^{k} \alpha_i, \qquad \alpha! = \prod_{i=1}^{k} \alpha_i!.$$

The space of polynomials of total degree at most n in k variables is

$$\Pi_n^k = \Big\{ \sum_{|\alpha| \leq n} c_\alpha x^\alpha \Big\}, \qquad \dim \Pi_n^k = \binom{n+k}{k} =: N.$$

Let $\mathcal{L}_m = \{L_1, \dots, L_m\}$ be a collection of $(k-1)$-dimensional hyperplanes in $\mathbb{R}^k$.

Denote by $\mathbb{I}_k^m$ the set of all strictly increasing k-tuples from $\{1, \dots, m\}$:

$$\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{I}_k^m \iff 1 \leq \alpha_1 < \dots < \alpha_k \leq m.$$

Definition. *The family $\mathcal{L}_m$ is in general position if:*
(i) the intersection of any k distinct hyperplanes is a single point;
(ii) the intersection of any $k+1$ distinct hyperplanes is empty.
If only condition (i) is satisfied, we say that $\mathcal{L}_m$ is admissible.

The intersection points are denoted

$$x_\alpha := L_{\alpha_1} \cap \dots \cap L_{\alpha_k}, \qquad \alpha \in \mathbb{I}_k^m.$$

Note that condition (ii) means that all points x_α are distinct.

Assume now that $\mathcal{L}_{n+k} = \{L_1, \dots, L_{n+k}\}$ is in general position. Then there are exactly $N = \binom{n+k}{k}$ distinct intersection points. To simplify notation, we assume that the hyperplane L_i is given by a linear equation $L_i(x) = 0$, that is, $L_i \in \Pi_1^k$.

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Theorem 1. *[1]. For any data* $\{c_\alpha : \alpha \in \mathbb{I}_k^{n+k}\}$ *there exists a unique* $p \in \Pi_n^k$ *such that*

$$p(x_\alpha) = c_\alpha \quad \forall \alpha \in \mathbb{I}_k^{n+k}. \tag{1}$$

Note that the fundamental polynomial of x_α is

$$p_\alpha^\star(x) = \frac{1}{A_\alpha} \prod_{\substack{i=1 \\ i \notin \alpha}}^{n+k} L_i(x),$$

where A_α is the normalizing constant, so that $p_\alpha^\star(x_\alpha) = 1$.

Then the Lagrange formula gives the polynomial satisfying (1):

$$p(x) = \sum_{\alpha \in \mathbb{I}_k^{n+k}} c_\alpha \, p_\alpha^\star(x). \tag{2}$$

Hermite Interpolation. Now assume that $\mathcal{L}_{n+k}$ is admissible only. Let

$$\mathcal{X} = \{x^{(1)}, \ldots, x^{(s)}\}$$

be the set of all distinct intersection points of the hyperplanes of $\mathcal{L}_{n+k}$.

Define *multiplicity* of $x^{(i)}$ as

$$m_i = \#\{j : x^{(i)} \in L_j,\ 1 \le j \le n+k\} - k + 1.$$

Denote for $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{Z}_{\ge 0}^k$

$$D^\alpha f = \frac{\partial^{|\alpha|}}{\partial {x_1}^{\alpha_1} \cdots \partial {x_k}^{\alpha_k}} f.$$

The Hermite interpolation data consists of the values of all partial derivatives up to the order $m_i - 1$ of a polynomial at each point $x^{(i)}$.

We say a point $x^{(i)}$ is *simple*, if its multiplicity equals to 1. Note that at a simple point only the value of a polynomial is interpolated.

Below we present the Hermite multivariate polynomial interpolation in the framework of the Chung–Yao approach.

Theorem 2. *(Th. 3, [2]). For any data* $\{c_i^\alpha : 1 \le i \le s,\ |\alpha| \le m_i - 1\}$ *there exists a unique* $p \in \Pi_n^k$ *satisfying*

$$D^\alpha p(x^{(i)}) = c_i^\alpha \quad \forall 1 \le i \le s,\ \forall |\alpha| \le m_i - 1. \tag{3}$$

In the Lagrange case of Chung–Yao interpolation we have formula (2) for the interpolation polynomial satisfying the conditions (1). To my knowledge, in the case of Hermite, there was previously no formula for the interpolation polynomial satisfying the conditions (3).

Next we discuss the problem of finding a formula for the latter polynomial.

New Hermite Multivariate Interpolation Formula. Let f be sufficiently smooth. The Taylor polynomial of total degree m for f at $c \in \mathbb{R}^k$ is

$$\mathcal{T}_{f,c,m}(x) = \sum_{|\alpha| \le m} \frac{D^\alpha f(c)}{\alpha!} (x - c)^\alpha.$$

It satisfies the conditions

$$D^\alpha \mathcal{T}_{f,c,m}(c) = D^\alpha f(c) \quad \forall |\alpha| \le m. \tag{4}$$

Define the global vanishing polynomial

$$\phi(x) = \prod_{j=1}^{n+k} L_j(x)$$

and the corresponding polynomial vanishing outside the point $x^{(i)}$

$$\phi_i(x) = \prod_{\substack{j=1 \\ x^{(i)} \notin L_j}}^{n+k} L_j(x).$$

Let $p_f \in \Pi_n^k$ be the unique Hermite interpolant of f, i.e.

$$D^\alpha p_f(x^{(i)}) = D^\alpha f(x^{(i)}) \quad \forall 1 \le i \le s,\ \forall |\alpha| \le m_i - 1.$$

Proposition. *Let $\mathcal{L}_{n+k}$ be admissible. Then the following explicit formula holds:*

$$p_f(x) = \sum_{i=1}^{s} \phi_i(x) \cdot \mathcal{T}_{f/\phi_i, x^{(i)}, m_i - 1}(x). \tag{5}$$

Let us call this Lagrange–Taylor formula.

Proof. It suffices to show that each fixed term

$$p_i(x) := \phi_i(x) \cdot \mathcal{T}_i, \text{ where } \mathcal{T}_i := \mathcal{T}_{f/\phi_i, x^{(i)}, m_i - 1}(x)$$

satisfies the following two groups of conditions.

1. ***Vanishing at other points*** $x^{(r)}$***, $r \neq i$, up to total order*** $m_r - 1$**.** Indeed, exactly $m_r + k - 1$ hyperplanes from $\mathcal{L}$ pass through $x^{(r)}$. At most $k-1$ of them can also pass through $x^{(i)}$ (since otherwise $x^{(r)} = x^{(i)}$). Therefore, at least m_r linear factors of ϕ_i vanish at $x^{(r)}$. So all derivatives of p_i up to order $m_r - 1$ vanish at $x^{(r)}$.

2. ***Correct reproduction at*** $x^{(i)}$ ***up to total order*** $m_i - 1$**.** By the multivariate Leibniz rule,

$$D^\alpha p_i(x^{(i)}) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \left(D^\beta \phi_i\right)(x^{(i)}) \cdot \left(D^{\alpha-\beta} \mathcal{T}_i\right)(x^{(i)}). \tag{6}$$

Since $|\alpha| \le m_i - 1$ and $\mathcal{T}_i$ reproduces all derivatives of f/ϕ_i up to order $m_i - 1$ at $x^{(i)}$, we have

$$D^{\alpha-\beta} \mathcal{T}_i(x^{(i)}) = D^{\alpha-\beta} \left(\frac{f}{\phi_i}\right)(x^{(i)})$$

for every term in the sum (6). Therefore,

$$D^\alpha p_i(x^{(i)}) = D^\alpha \left(\phi_i \cdot \frac{f}{\phi_i}\right)(x^{(i)}) = D^\alpha f(x^{(i)}).$$

This completes the Proof. □

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Հ. Ա. ՀԱԿՈԲՅԱՆ

ՀԵՐՄԻԹԻ ԲԱԶՄԱՉԱՓ ՄԻՋԱՐԿԱՅԻՆ ԲԱՆԱՁԵՎ

Մենք ներկայացնում ենք Հերմիթի բազմաչափ միջարկման խնդրի նոր բանաձև Չանգ–Յաոյի մոտեցման շրջանակներում:

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