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Mathematics

ON SUM EDGE-COLORINGS OF SOME REGULAR GRAPHS

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A sum edge-coloring of a graph is an assignment of positive integers to the edges of the graph, so that adjacent edges correspond to different numbers (colors) and the sum of the numbers on all the edges is minimum possible. This minimum possible sum is called the edge-chromatic sum of the graph, and the minimal number of colors needed for a sum edge-coloring is called the edge-strength of the graph. In this paper, we give the exact values of the edge-chromatic sums and edge-strengths for cycle powers and generalized cycles.

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Introduction. Concepts used in this paper that are not defined should be assumed to match the terminology of West from [1]. The graphs considered in this paper are finite, undirected, and simple.

An assignment α of positive integers to the edges of the graph G is called a proper edge-coloring, if all adjacent edges e and e' satisfy $\alpha(e) \neq \alpha(e')$. The minimum number of colors needed to construct a proper edge-coloring is called the edge-chromatic index of the graph G and is denoted by $\chi'(G)$. For proper edge-coloring α we denote $\sum'(G,\alpha) = \sum\limits_{e\in E(G)} \alpha(e)$. The minimum possible $\sum'(G,\alpha)$ among all proper edge-colorings of G is called the *edge-chromatic sum* of the graph G and is denoted by $\sum'(G)$. Proper edge-colorings α_i, for which $\sum'(G,\alpha_i) = \sum'(G)$, are called sum edge-colorings, and the minimum number of colors needed to construct a sum edge-coloring for a graph G is called the *edge-strength* of G and is denoted by $s'(G)$. Obviously, $s'(G) \geq \chi'(G)$.

For positive integers n and m, where $n \geq 3$, we define a generalized cycle $C_n(m)$ as a graph with the vertex set $V(C_n(m)) = \{V^1, V^2, ..., V^n\}$, where $V^i = \{v^i_j \mid j \in 1,2,...,m\}$ for $i \in 1,2,...,n$, and the edge set $E(C_n(m)) = \{uv \mid u \in V^i, v \in V^{i+1}, i = 1,2,...,n\}$. Here and henceforth, by V^{n+1} we will mean V^1. An example of a generalized cycle is illustrated in Fig. 1.

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For positive integers n and m, where $n \geq 3$ and $2m+1 \leq n$, we define a cycle power graph C_n^m as a graph with the vertex set $V(C_n^m) = \{v_1, v_2, \ldots, v_n\}$ and the two vertices v_i and v_j $(i < j)$ of which are connected by an edge if and only if $j - i \leq m$ or $j - i \geq n - m$. An example of a cycle power is illustrated in Fig. 2.

The sum edge-coloring problem was introduced by Bar-Noy et al. in 1998 [2]. The problem is shown to be NP-hard [2], even for very specific classes of graphs such as regular graphs [3] and bipartite graphs with maximum degree 3 [4] and even for some more specific class of graphs within the latter [5]. There are also known approximation algorithms, for example, there is a 2-approximation algorithm for general graphs [2], $\frac{11}{8}$-approximation algorithm for regular graphs [5]. In [5], an upper bound of the edge-chromatic sum of some split graphs is also given.

There is a foundational theorem in graph coloring theory, proved by Vizing, which states that for any graph G, $\Delta(G) \leq \chi'(G) \leq \Delta(G)+1$, and according to that distinction, graphs for which $\chi'(G) = \Delta(G)$ are called Class 1 graphs, while the graphs for which $\chi'(G) = \Delta(G)+1$ are called Class 2 graphs.

For the edge-strength parameter of graphs, a theorem similar to Vizing's is proven by Hajiabolhassan:

Theorem 1. *[6]. For any graph G, $\Delta(G) \leq s'(G) \leq \Delta(G)+1$.*

Salavatipour [3] proved that deciding whether $s'(G) = \Delta(G)$ or $s'(G) = \Delta(G)+1$ is NP-complete even for regular graphs.

Note that simple cycles C_n are regular graphs, for which the edge-chromatic sum can be easily computed and the edge-strength depends on the parity of n. Parker [7] defined regular graphs $C_n(m)$ that are generalizations of simple cycles, and provided the exact values of $\chi'(C_n(m))$:

Theorem 2. *[7]. The generalized cycle $C_n(m)$ is Class 1 if and only if nm is even.*

Another family of regular graphs, for which the chromatic index is known, are powers of cycles:

Theorem 3. *[8]. The graph C_n^m is Class 1 if and only if n is even.*

The current work gives the exact values of the edge-chromatic sum and edge-strength parameters for the graphs $C_n(m)$ and C_n^m.

We will also use the following lemma and theorem:

Lemma. *[9]. For a graph G with $s'(G) \geq 2$ and any $k \in \mathbb{N}$ that satisfies $2 \leq k \leq s'(G)$, we have:*

$$\sum{}'(G) \geq k\left(|E(G)| - \frac{k-1}{2}\left\lfloor\frac{|V(G)|}{2}\right\rfloor\right) + \frac{(s'(G)-k)(s'(G)-k+1)}{2}.$$

For referencing the following theorem, we define a complete tripartite graph as follows: for any natural numbers n, m, and l, we define the complete tripartite graph $K_{n,m,l}$ as a graph with the vertex set

$$V(K_{n,m,l}) = \{v_1, v_2, ..., v_n, u_1, u_2, ..., u_m, w_1, w_2, ..., w_l\}$$

and the edge set

$$E(K_{n,m,l}) = \{v_i u_j : 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{v_i w_j : 1 \leq i \leq n, 1 \leq j \leq l\}$$
$$\cup \{u_i w_j : 1 \leq i \leq m, 1 \leq j \leq l\}.$$

Theorem 4. *[9]. For any odd $n \in \mathbb{N}$,*

$$\sum{}'(K_{n,n,n}) = \frac{n(2n+1)(3n+1)}{2}.$$

Moreover, a sum edge-coloring β_n does not use the color $2n+1$ in edges $u_i w_j$ $(1 \leq i, j \leq n)$.

Generalized Cycles.

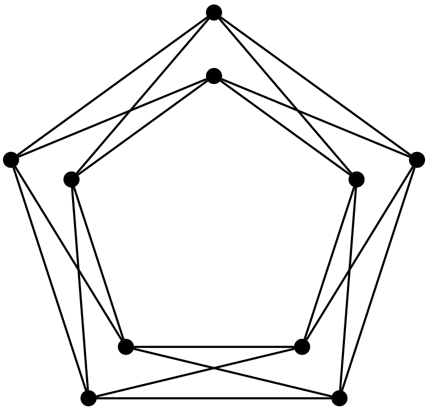

Fig. 1. Generalized cycle $C_5(2)$.

Here we give a theorem that gives the exact values of the edge-chromatic sum and edge-strength of generalized cycles.

Theorem 5. *For any $n \geq 3$, $m \geq 1$, we have:*

$$\sum{}'(C_n(m)) = \begin{cases} \dfrac{nm^2(2m+1)}{2}, & \text{if } nm \text{ is even}, \\ \dfrac{m(nm+1)(2m+1)}{2}, & \text{otherwise}, \end{cases}$$

and

$$s'(C_n(m)) = \begin{cases} 2m, & \text{if } nm \text{ is even}, \\ 2m+1, & \text{otherwise}. \end{cases}$$

Proof. If nm is even, then from Theorem 2 we have that the graph is Class 1, so there is a proper edge $\Delta(C_n(m)) = 2m$-coloring α of $C_n(m)$. Since each vertex is incident to exactly $2m$ edges, all $2m$ colors appear on edges incident to each vertex. This means that each color is used $\frac{V(C_n(m))}{2} = \frac{nm}{2}$ times. Hence, $\sum'(C_n(m), \alpha) = nm^3 + \frac{nm^2}{2}$. On the other hand, from Lemma $\sum'(C_n(m)) \geq nm^3 + \frac{nm^2}{2}$. Hence, it remains the case, where both n and m are odd.

In this case, Theorems 1 and 2 give us that $s'(C_n(m)) = 2m+1$. If we put $k = 2m+1$ in Lemma, we obtain $\sum'(C_n(m)) \geq \frac{m(nm+1)(2m+1)}{2}$. In order to prove that $\sum'(C_n(m)) \leq \frac{m(nm+1)(2m+1)}{2}$, we construct a corresponding sum edge $2m+1$-coloring. We do an induction on n and construct a coloring α_n that satisfies the following properties:

1. $\alpha_n(v_i^1 v_j^n) < 2m+1$ for each $1 \leq i,\ j \leq m$;
2. Each color from 1 to $2m$ is used $\frac{nm-1}{2}$ times;
3. And the number of edges colored by $2m+1$ is m.

Let us call this property the (a) property. For $n = 3$, we use the coloring β_n from Theorem 4 and construct α_3 as follows: for each $1 \leq i,\ j \leq m$:

$$\alpha_3(v_i^1 v_j^2) = \beta_n(u_i v_j), \alpha_3(v_i^2 v_j^3) = \beta_n(v_i w_j), \alpha_3(v_i^1 v_j^3) = \beta_n(u_i w_j).$$

Now we assume that a proper edge $2m+1$-coloring α_{n-2} with the property (a) is already constructed. We construct α_n as follows: first we color the following edges: for each $1 \leq j,\ l \leq m$ and $1 \leq i \leq n-2$, let $\alpha_n(v_j^i v_l^{i+1}) = \alpha_{n-2}(v_j^i v_l^{i+1})$. Let $G = (W \cup U, E)$ be a bipartite graph where $W = V^1 \cup V^{n-1} \subseteq V(C_n(m))$, $U = \{u_1, u_2, ..., u_{2m}\}$ and E connects the vertices v_i^1 of V^1 to a vertex u_j, if there is an edge adjacent to v_i^{n-1} in $C_n(m)$ that is already colored with the color j, and connects the vertices v_i^{n-1} of V^{n-1} to a vertex u_j, if there is no edge adjacent to v_i^{n-1} in $C_n(m)$ that is already colored with the color j. Note that since α_{n-2} satisfies the property (a), there is no color $2m+1$ on the edges adjacent to the vertices of V^{n-1} in α_n. Also, for each color i in $1, 2, ..., 2m$ and for each j in $1, 2, ..., m$, the vertex u_i is either connected to v_j^1 or to v_j^{n-1}. So the graph G is a bipartite m-regular graph. If we denote a proper edge m-coloring of G by γ, we can construct the remaining colors of α_n as follows: for each $1 \leq i \leq m$, $p \in \{1, n-1\}$, and $1 \leq j \leq 2m$, if $\gamma(v_i^p u_j) = k$, let $\alpha_n(v_i^p v_k^n) = j$. It is not hard to see that the resulting coloring is a proper $(2m+1)$-coloring and satisfies the property (a). To complete the Proof, let us see that the coloring contains each color from 1 to $2m$ exactly $\frac{nm-1}{2}$ times and the color $2m+1$ exactly m times, so the total sum of colors matches the expected expression. □

Cycle Powers.

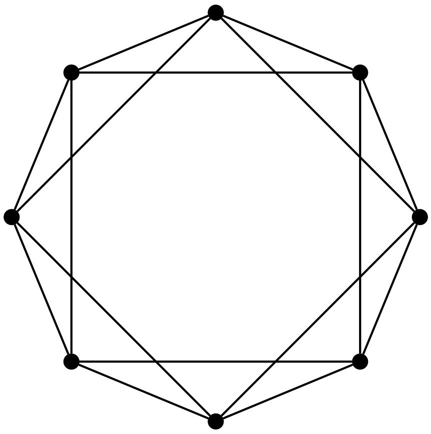

Fig. 2. Cycle Power C_8^2.

Theorem 6. *For any $n \geq 3$, $m \geq 1$, $2m+1 \leq n$, we have:*

$$\sum{}'(C_n^m) = \begin{cases} \dfrac{nm(2m+1)}{2}, & \text{if } n \text{ is even}, \\ \dfrac{m(n+1)(2m+1)}{2}, & \text{otherwise}, \end{cases}$$

and

$$s'(C_n^m) = \begin{cases} 2m, & \text{if } n \text{ is even}, \\ 2m+1, & \text{otherwise}. \end{cases}$$

Proof. If n is even, then Theorem 3 gives us that C_n^m admits a proper edge $2m$-coloring β. Similar to the previous theorem, since the degree of each vertex is $2m$, each color appears on the edges incident to each vertex. There are n vertices in total, so each color is used $\frac{n}{2}$ times. Hence $\sum{}'(C_n^m, \beta) = \frac{nm(2m+1)}{2}$. Lemma gives also that $\sum{}'(C_n^m) \geq \frac{nm(2m+1)}{2}$, which ends the proof for even n.

For odd n, Theorem 3 and Theorem 1 give us $s'(C_n^m) = 2m+1$ and Lemma gives us $\sum{}'(C_n^m) \geq \frac{m(n+1)(2m+1)}{2}$, so it only remains to show a proper edge coloring γ such that $\sum{}'(C_n^m, \gamma) = \frac{m(n+1)(2m+1)}{2}$.

Let us decompose $E(C_n^m)$ into sets $E_1 \cup E_2 \cup \cdots \cup E_m$, where $E_i = \{v_j v_l \mid 1 \leq j < l \leq n \text{ and } (l-j = i \text{ or } l-j = n-i)\}$ $(1 \leq i \leq m)$. Note that the set $E(C_n^m) \cap \left\{ v_i v_{n-i} \mid 1 \leq i \leq \frac{n-1}{2} \right\}$ is a matching of size m in C_n^m each edge of which lies in one of E_i. We color these edges with the color $2m+1$ in γ. Now we color the remaining edges from $E_m \cup E_{m-1}$ with 4 colors, the remaining edges from $E_{m-2} \cup E_{m-3}$ with 4 new colors, etc. until $E_2 \cup E_1$ with 4 new colors or E_1 with 2 new colors depending on the parity of m. If m is odd, we need to color E_1 with 2 colors, which is trivial to do since E_1 is a hamiltonian cycle in C_n^m one edge of which is already colored with $2m+1$ and the other edges can be colored with alternating colors to make the coloring proper. Thus, it remains to properly color the edges of $E_i \cup E_{i+1}$ (for certain i) with 4 colors considering one edge from E_i and one edge from E_{i+1} are already colored with $2m+1$. Note that if we do that, we will achieve $\sum'(C_n^m,\gamma) = \frac{m(n+1)(2m+1)}{2}$ since each color from 1 to $2m$ will be used $\frac{n-1}{2}$ times and the color $2m+1$will be used m times.

Now consider any E_i and E_{i+1}. If both E_i and E_{i+1} are hamiltonian cycles, we can color each of them with 2 colors as above and the problem is solved. Otherwise, as Bermond et al. showed in [10] in Proposition, the graph $E_i \cup E_{i+1}$ can be represented as a graph G, where $V(G) = \{(u_i,v_j)|\ 0 \le i < a,\ 0 \le j < k\}$ and $E(G) = \{(u_i,v_j)(u_i,v_{j+1})|0 \le i < a,\ 0 \le j < k\} \cup \{(u_i,v_j),(u_{i+1},v_j)|\ 0 \le i < a-1,\ 0 \le j < k\} \cup \{(u_{a-1},v_j)(u_0,v_{j+c})|\ 0 \le j < k\}$ (note that by $j+1$ and $j+c$ we mean a sum operation modulo k), where $a \ge 3$, $k \ge 3$, and $a \cdot k = n$. Moreover, the construction is not unique and the vertices $(u_0,v_0),(u_0,v_1)$, and (u_1,v_0) can be chosen in a way that $(u_0,v_0)(u_0,v_{k-1})$ is one of the edges colored $2m+1$, the other edge colored $2m+1$ is one of $\{(u_i,v_j)(u_{i+1},v_j)|\ 0 \le i < a-1,\ 0 \le j < k\}$, and the graph G can be constructed upon this choice. First, we provide the coloring γ on an example when this edge is $(u_0,v_{k-2})(u_1,v_{k-2})$. The other cases will be easy to show after a few shifts on γ. We construct the coloring γ as follows (we denote the 4 colors by $1,2,3,4$, and the color $2m+1$ by 5 for simplicity):

1) for any $0 \le j \le \frac{k-3}{2}$, let
$\gamma((u_0,v_{2j})(u_0,v_{2j+1})) = 2$ and $\gamma((u_0,v_{2j+1})(u_0,v_{2j+2})) = 3$;
2) for any $1 \le i \le a-1, 0 \le j \le \frac{k-3}{2}$, let $\gamma((u_i,v_{2j})(u_i,v_{2j+1})) = 3$;
3) for any $1 \le i \le a-2, 0 \le j \le \frac{k-5}{2}$, let $\gamma((u_i,v_{2j+1})(u_i,v_{2j+2})) = 4$;
4) for any $0 \le j \le \frac{k-5}{2}$, let $\gamma((u_{a-1},v_{2j}))(u_{a-1},v_{2j+1})) = 1$;
5) for any $1 \le i < a$, let $\gamma((u_i,v_{k-2})(u_i,v_{k-1})) = 2$;
6) $\gamma((u_0,v_0)(u_0,v_{k-1})) = 5$, $\gamma((u_{a-1},v_0)(u_{a-1},v_{k-1})) = 1$,
$\gamma((u_0,v_{k-2})(u_1,v_{k-2})) = 5$, $\gamma((u_0,v_{k-1})(u_1,v_{k-1})) = 1$,
$\gamma((u_1,v_{k-2})(u_2,v_{k-2})) = 1$, and $\gamma((u_1,v_{k-1})(u_2,v_{k-1})) = 3$;
7) for any $1 \le i \le a-2$, let $\gamma((u_i,v_0)(u_i,v_{k-1})) = 4$;
8) for any $0 \le i \le \frac{a-3}{2}, 0 \le j \le k-3$, let
$\gamma((u_{2i},v_j)(u_{2i+1},v_j)) = 1$ and $\gamma((u_{2i+1},v_j)(u_{2i+2},v_j)) = 2$;
9) for any $1 \le i \le \frac{a-3}{2}$, let
$\gamma((u_{2i},v_{k-2})(u_{2i+1},v_{k-2})) = 4$, $\gamma((u_{2i},v_{k-1})(u_{2i+1},v_{k-1})) = 1$,
$\gamma((u_{2i+1},v_{k-2})(u_{2i+2},v_{k-2})) = 1$, and $\gamma((u_{2i+1},v_{k-1})(u_{2i+2},v_{k-1})) = 3$;
10) for any $0 \le j \le k-3$, let $\gamma((u_{a-1},v_j)(u_0,v_{j+c})) = 4$.

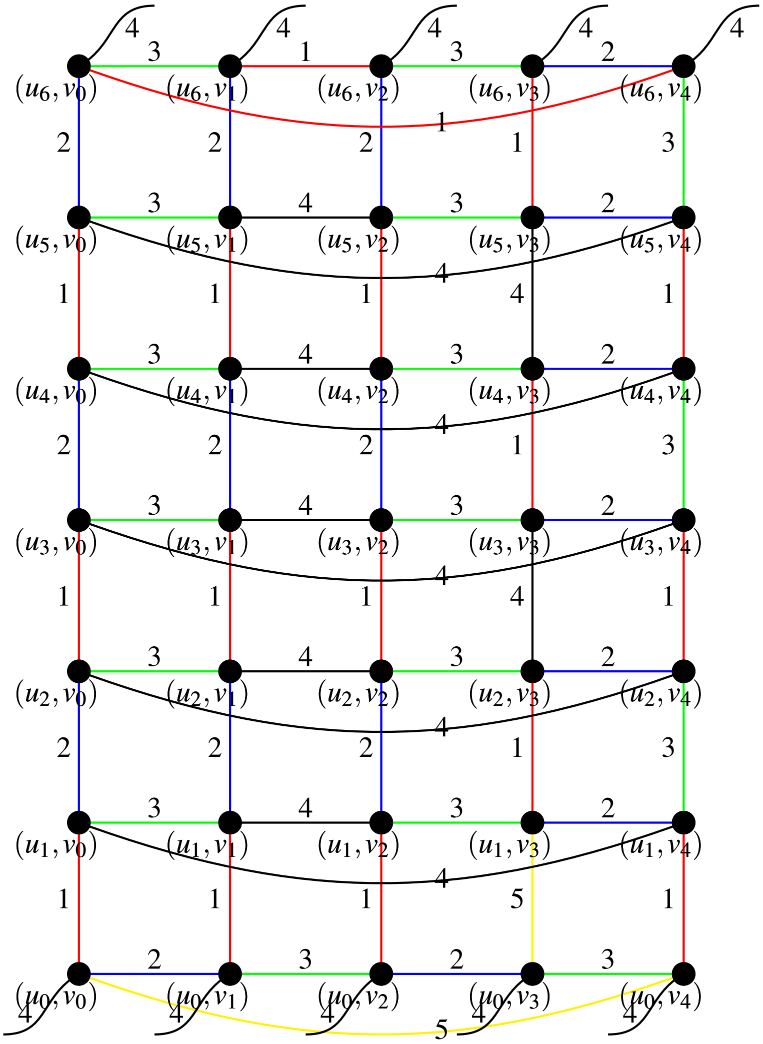

Fig. 3. The graph with $a=7$ and $k=5$.

If the second color $2m+1$ is not on $(u_0,v_{k-2})(u_1,v_{k-2})$, but is on some edge $(u_i,v_{k-2})(u_{i+1},v_{k-2})$, note that we can construct the coloring by taking the current coloring γ and alternatively recoloring the edges $(u_0,v_{k-2})(u_1,v_{k-2})$, $(u_1,v_{k-2})(u_2,v_{k-2})$, $\ldots,(u_{i-1},v_{k-2})(u_i,v_{k-2})$ with colors 1 and 4. On the other hand, if this edge is on $(u_0,v_j)(u_1,v_j)$ that $k-2-j=d>0$, we can construct the coloring γ' by assigning $\gamma'((u_i,v_l)(u_p,v_q))=\gamma((u_{i+d},v_l)(u_{p+d},v_q))$ to each edge except $(u_0,v_j)(u_0,v_{j+1})$, which we assign the same color as in γ. The $i+d$ and $p+d$ operations are modulo a. Similarly, we can construct the coloring for each case for the appearance of the edge colored with $2m+1$ by applying the two shifting methods. $\square$

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Հ. Վ. ՄԻՔԱԵԼՅԱՆ

ՈՐՈՇ ՀԱՄԱՍԵՌ ԳՐԱՖՆԵՐԻ ԳՈՒՄԱՐԱՅԻՆ ԿՈՂԱՅԻՆ ՆԵՐԿՈՒՄՆԵՐՆԵՐԻ ՄԱՍԻՆ

Գրաֆի գումարային կողային ներկումը նրա կողերի համապատասխանեցումն է բնական թվերին այնպես, որ հարևան կողերը համապատասխանում են տարբեր թվերի (գույների) և բոլոր կողերի թվերի գումարը նվազագույնն է: Այդ նվազագույն գումարը կոչվում է գրաֆի կողային քրոմատիկ գումար, իսկ գումարային կողային ներկման համար անհրաժեշտ գույների նվազագույն քանակը կոչվում է գրաֆի կողային ամրություն: Այս աշխատանքում տրվել են ցիկլերի աստիճանների և ընդհանրացված ցիկլերի կողային քրոմատիկ գումարի և կողային ամրության ճշգրիտ արժեքները:

Г. В. МИКАЕЛЯН

О СУММАРНЫХ РЕБЕРНЫХ РАСКРАСКАХ НЕКОТОРЫХ СТАНДАРТНЫХ ГРАФОВ

Суммарная реберная раскраска графа – это отображение его ребер на положительные целые числа таким образом, чтобы смежным ребрам соответствовали разные числа (цвета), а сумма чисел всех ребер была минимальна. Эта минимальная сумма называется реберно-хроматической суммой графа, а минимальное количество цветов, необходимое для суммарной реберной раскраски, называется реберной мощностью графа. В данной работе мы приводим точные значения реберно-хроматических сумм и реберной мощности для степеней циклов и обобщенных циклов.